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Eschenburg space as gravity dual of flavored $\mathcal{N} = 4$ Chern-Simons-matter theory

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Abstract: We find a 3D flavored $\mathcal{N}$=4 Chern-Simons-matter theory, a kind of $\mathcal{N}$=3 SCFT, has a gravity dual $AdS_4 \times \mathcal{M}_7(t_1, t_2, t_3)$ where three coprime parameters can be read off according to the number and charge of 5-branes in the dual Type IIB string setup. Because $\mathcal{M}_7(t_1, t_2, t_3)$ has been known in literature as Eschenburg space, we exploit some of its properties to examine the correspondence between two sides.

Keywords: AdS-CFT Correspondence, Differential and Algebraic Geometry, M-Theory

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Contents

1 Introduction

The program towards studying gauge/gravity correspondence in the context of $\mathrm{AdS}_4/\mathrm{CFT}_3$ [1] becomes concrete owing to the pioneering work [2] by Aharony, Bergman, Jafferis and Maldacena last year. They found that constructing a much higher supersymmetric conformal field theory (SCFT) of Chern-Simons-matter (CSM) type is possible due to an elliptic brane setup in Type IIB string theory. Through T-duality and M-theory lift, one obtains N M2-branes filling (012) transverse to a 8D cone: Cone($\mathcal{B}_7$) ($\mathcal{B}_7 = S^7/\mathbf{Z}_k$) along (345678910). The corresponding gravity dual is thus a solution of 11D supergravity $AdS_4 \times \mathcal{B}_7$ after N M2-branes backreact.

Later on, generalizing their idea to yield elliptic $\mathcal{N} \geq 3$ SCFTs is explained in [3, 4] by attaching various kinds of $(1, k_i)$5-branes on a stack of circular D3-branes. The resulting field theory at infra-red (IR) fixed point ($g_{\mathrm{YM}} \to \infty$) is still of quiver CSM type with product gauge group $\prod_I \mathrm{U}(N)_I$. Its Lagrangian is so rigid, i.e. CS level k_I w.r.t. I-th gauge factor is determined by two adjacent D5-brane charges $k_I = k_i - k_{i-1}$ [5, 6], while the superpotential is obtained by integrating out non-dynamical massive adjoint $\Phi_I(\subset$ vector multiplet) coupled to hypermultiplets in a typical manner.

Among many kinds of elliptic $\mathcal{N} \geq 3$ SCFTs, we will focus on a specific type of $\mathcal{N}$=4 SCFT which is constructed via IIB N circular D3- (0126), p NS5- (012345) and q $(1,k)$5- $(012[3,7]_\theta[4,8]_\theta[5,9]_\theta)$ branes.[1] Its 11D gravity dual $AdS_4 \times \mathcal{M}_7$ parameterized by (k,p,q) is explicitly known [3, 10] and this is the main reason why we study this kind of SCFT here.

In this note, motivated by works on the flavored ABJM theory [7–9], we construct a new $\mathcal{N}$=3 SCFT by adding N_F massless fundamental flavors and study its gravity dual. From Type IIB picture, adding flavor corresponds to further attaching N_F D5-branes (012789) on the circle x^6 and results in a less supersymmetric $\mathcal{N}$=3 SCFT. This construction is by definition an elliptic one, so in M-theory to have N M2-branes probing a 8D cone may thus be expected due to conformality.

We find that this turns out to be true and the dual geometry is now $AdS_4 \times \tilde{\mathcal{M}}_7$ parameterized by three natural numbers $(t_1,t_2,t_3) = (qN_F, pN_F, kpq)$ without any common factor. In fact, many properties of $\tilde{\mathcal{M}}_7(t_1,t_2,t_3)$ (modulo common factor) known as Eschenburg space [11] have been explored by mathematicians [12–14]. For example, Cone($\tilde{\mathcal{M}}_7$) is Ricci-flat with special Sp(2) holonomy. Namely, it is hyperKähler and the base $\tilde{\mathcal{M}}_7$ must be tri-Sasakian (which preserves a fraction 3/16 of 32 SUSY). Moreover, the cone is available through applying a hyperKähler quotient to a 3D (flat) quaternionic space, say, $\mathbf{H}^3///\mathrm{U}(1)_Q \equiv \boldsymbol{\mu}_Q^{-1}(0)/\mathrm{U}(1)_Q$[2] and (t_1,t_2,t_3) stands for the underlying $\mathrm{U}(1)_Q$ charge assignment respectively for three quaternions.[3]

In order to understand relations better between the three, say, flavored $\mathcal{N}$=4 CSM, $AdS_4 \times \tilde{\mathcal{M}}_7$ M-theory dual and IIA gravity dual of $\mathcal{N}$=4 CSM with probe flavor branes embedded, we adopt the viewpoint similar to [15, 16]. That is, we compute the entropy of the three. On the field theory side, we take large N zero-coupling limit and compactify $R^{1,2}$ on $S^1 \times S^2$. Its partition function is finally expressed in terms of an unitary matrix model which is exactly solvable. A similar formulation using a matrix model allows us to evaluate $\mathcal{N}$=3 superconformal index to which we are able to compare Kaluza-Klein (KK) analysis on $\tilde{\mathcal{M}}_7$.

On the geometry side, we are led to compute the volume of 7-cycle of Eschenburg space by taking advantage of a formula given in [17]. Also, the on-shell action of IIA probe

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[1] θ (twisted angle) and $g^2_{\mathrm{YM}}k/4\pi$ (adjoint mass) are related to each other by (L: segment length on x^6)

$$\frac{\tan\theta}{L} = g^2_{\mathrm{YM}}k, \qquad \frac{1}{g^2_{\mathrm{YM}}} = \frac{L}{g_s}.$$

Taking IR limit implies naturally a strongly coupled M-theory picture.

[2] Certainly, one can solve D- and F-term conditions to see that M2-brane moduli space is $\mathbf{Sym}^N(\mathbb{M})$ with $\mathbb{M} \sim \mathbf{H}^3///\ \mathrm{Ker}(\beta)$ and

$$\beta : \mathrm{U}(1)^3 \to \mathrm{U}(1)^2, \qquad \beta = \begin{pmatrix} p & q & 0 \\ 0 & kq & N_F \end{pmatrix}. \tag{1.1}$$

The equivalence between two descriptions seems straightforward because three moment maps $\boldsymbol{\mu}_Q = \sum_{i=1}^3 t_i \boldsymbol{\mu}_i$, $p\boldsymbol{\mu}_1 + q\boldsymbol{\mu}_2$ and $kq\boldsymbol{\mu}_2 + N_F\boldsymbol{\mu}_3$ are linearly independent.

[3] Note that the amount of its isometry is essentially $\mathrm{SU}(2)_R \times \mathrm{U}(1)^2$ but $\mathrm{U}(1)^2$ gets enhanced to $\mathrm{SU}(2) \times \mathrm{U}(1)$ (SU(3)) if two (all) of three t's coincide.

D6-branes is taken care of. As pointed out in [8], the correct embedding of D6-branes can be found by performing a further hyperKähler quotient to Cone($\mathcal{M}_7$). One obtains a 4D Taub-NUT space thereof over which flavor probes should wrap after doing KK reduction to IIA theory. In addition, we consider 5-cycles among $\tilde{\mathcal{M}}_7$ because M5-branes wrapped over them correspond to baryonic operators in the field theory.

This note is organized as follows. We begin with discussing how to obtain a 3D $\mathcal{N}$=3 SCFT from an elliptic $\mathcal{N}$=4 one. Then in section 3 and section 4, we study the entropy of our underlying $\mathcal{N}$=3 SCFT by means of both field theory and gravity approaches. An $\mathcal{N}$=3 superconformal index is computed in section 5 and comments about baryonic operators are in section 6. Finally, a conclusion is drawn. Appendices about Taub-NUT space in M-theory and mesonic operators are attached.

2 Adding flavors to 3D $\mathcal{N}=4$ SCFT

In section 2.1, we shortly review some aspects about $\mathcal{N}$=4 Chern-Simons-matter theory [3, 18, 19]. In section 2.2, by adding massless flavors to it, a new $\mathcal{N}$=3 SCFT is constructed. We observe that this $\mathcal{N}$=3 Lagrangian requires naturally a 8D hyperKähler internal space.

2.1 $\mathcal{N}=4$ SCFT

In order to obtain a desired $\mathcal{N}$=4 SCFT at IR fixed point, one begins with an ultra-violet (UV) Lagrangian $\mathcal{L}^{\text{UV}}$ containing $\mathcal{N}$=4 (V^I,Φ^I) vector-, (A^I,B^I) hyper-, and (A^J,B^J) twisted hyper-multiplets where I labels the gauge factor. $\mathcal{L}^{\text{UV}}$ can be read off from the corresponding IIB brane configuration (or quiver diagram) considered in section 1.[4] Due to CS terms induced, vector multiplets acquire mass $\sim k_I g_{\text{YM}}^2/4\pi$ and at low-energy limit $(g_{\text{YM}}\to\infty)$ kinetic terms of them proportional to $1/g_{\text{YM}}^2$ all decouple except for CS terms which do not depend on g_{YM}. The remaining non-dynamical adjoint Φ^I in F-terms or real scalar $\sigma^I\subset V^I$ in D-terms will be integrated out later. As a result, one arrives at a 3D $\mathcal{N}$=4 CSM theory with $\mathcal{L}^{\text{IR}}_{\text{bos}}=\mathcal{L}_{\text{CS}}+\mathcal{L}_{\text{hyper}}+\mathcal{L}_{\text{pot}}$ where (bosonic part only)

$$\begin{aligned}\mathcal{L}_{\text{hyper}} &= \text{Tr}\sum_I\int d^3xd^4\theta\,\left(\overline{A}^I e^{2V^I}A^I e^{-2V^{I+1}}+B^I e^{-2V^I}\overline{B}^I e^{2V^{I+1}}\right),\\ \mathcal{L}_{\text{pot}} &= \text{Tr}\sum_I\frac{1}{k_I}\int d^3xd^2\theta\,\left(B^IA^I-A^{I-1}B^{I-1}\right)^2+c.c..\end{aligned}$$

Unlike $\mathcal{N}$=3 Chern-Simons-Yang-Mills (CSYM) theory obtained by adding CS terms to $\mathcal{N}$=4 YM one [20, 21], that YM terms decouple here, on the contrary, doubles the amount of SUSY. The $\text{SO}(4)_R$ R-symmetry arises from $\text{SU}(2)_t\times\text{SU}(2)_{\text{unt}}$ rotating (A^I,B^I) and (A^J,B^J) which are massless open string modes across $(1,k)$5- and NS5-branes, respectively.[5] With the baryonic $\text{U}(1)_b$ and diagonal $\text{U}(1)_d$, these as a whole agree precisely with the isometry $\left(\text{SU}(2)\times\text{U}(1)\right)^2$ of its moduli space $\left(\mathbf{C}^2/\mathbf{Z}_p\times\mathbf{C}^2/\mathbf{Z}_q\right)/\mathbf{Z}_k$ [3]. The global symmetry of $\mathcal{N}$=4 CSM theory is summarized in table 1.

[4] In fact, there are some thoughts in dealing with $p\neq q$ as shown in [19] for zero CS levels, but we will ignore these subtleties.

[5] Note that $A^I\supset(h^I_\alpha,\psi^I_{\dot\alpha})$, $B^I\supset\left(\tilde h^I_\alpha,\tilde\psi^I_{\dot\alpha}\right)$ and $A^J\supset(h^J_{\dot\alpha},\psi^J_\alpha)$, $B^J\supset\left(\tilde h^J_{\dot\alpha},\tilde\psi^J_\alpha\right)$ where α $(\dot\alpha)$ denotes the spinor index of SU(2).

	$\mathrm{U}(N)_{I-1}$	$\mathrm{U}(N)_I$	$\mathrm{SU}(2)_R$	$\mathrm{SU}(2)_R$	$\mathrm{U}(1)_b$	$\mathrm{U}(1)_d$
h^I_α	$\mathbf{N}$	$\overline{\mathbf{N}}$	$\mathbf{2}$	$\mathbf{1}$	1	1
$h^I_{\dot\alpha}$	$\mathbf{N}$	$\overline{\mathbf{N}}$	$\mathbf{1}$	$\mathbf{2}$	-1	1

Table 1. The global symmetry of $\mathcal{N}=4$ CSM.

In summary, in IR limit the action consists of only CS terms, kinetic terms of hypermultiplets and a suitable superpotential. A comment is as follows. The structure of Lagrangian is quiet simple and one can think that CS terms impose a special kind of *gauging* of $(n-2)$ out of n U(1) factors,[6] except for an overall diagonal U(1) and the dual photon $\mathcal{A}_\mu = \sum_I k_I A^I_\mu$. Note that the remnant of $\mathcal{A}_\mu$ is some discrete gauge symmetry.

Due to the $(n-2)$ *gauging*, a direct observation is that the maximal global symmetry SO(8) of moduli space is broken to $\mathcal{G}\times \mathrm{U}(1)^2$, where $\mathrm{U}(1)^2$ correspond to ungauged ones. To determine $\mathcal{G}$ relies on knowing the complete moduli space. In $\mathcal{N}$=6 ABJM case, the diagonal U(1) gets included in the $\mathrm{SU}(4)_R$ R-symmetry. In ellptic $\mathcal{N}$=4 models above, $\mathcal{G}=\mathrm{SO}(4)_R$ does not get mixed with two U(1)'s.

2.2 Adding flavors

Next, let us add massless flavors to $\mathcal{N}=4$ SCFT, i.e. to I-th gauge group N^I_F fundamental hypermultiplets $(Q^I,\tilde{Q}^I)$ of $(\mathbf{N},\overline{\mathbf{N}})$ with $\sum_I N^I_F = N_F$. This results in an additional D-term

$$\mathcal{L}_{\text{flavor}} = \mathrm{Tr}\sum_{\alpha,I}\int d^3x d^4\theta\ \left(\overline{Q}^I_\alpha e^{2V^I}Q^I_\alpha + \tilde{Q}^I_\alpha e^{-2V^I}\overline{\tilde{Q}}{}^I_\alpha\right), \qquad \alpha=1,\cdots,N^I_F$$

and

$$\mathcal{L}_{\text{pot}}\to\mathcal{L}'_{\text{pot}} = \mathrm{Tr}\sum_{\alpha,I}\frac{1}{k_I}\int d^3x d^2\theta\left(B^IA^I - A^{I-1}B^{I-1} + Q^{I-1}_\alpha\tilde{Q}^{I-1}_\alpha\right)^2 + c.c\ldots$$

The R-symmetry is now broken to $\mathrm{SO}(3)_R$ which is the diagonal $\mathrm{SU}(2)_d\subset\mathrm{SU}(2)_R\times\mathrm{SU}(2)_R\simeq\mathrm{SO}(4)_R$, while $\mathrm{U}(1)_b\times\mathrm{U}(1)_d$ stays unchanged. We find this is consistent with the amount of isometry of Eschenburg space (with three different t's) as advertised in footnote 2.

3 Entropy from field theory

Let us do a very simple counting of the degrees of freedom in $\mathcal{N}$=3 SCFT. This is carried out by computing the entropy of a dilute gas of massless states via statistical mechanics. The system is put in a box of size $V_2=L^2$ and momenta of massless states are quantized as $\vec{p}=2\pi\vec{n}/L$ ($\vec{n}\in\mathbf{Z}^2$).

$$S=-\frac{\partial F}{\partial T},\qquad F=-T\log Z = T\frac{V_2}{4\pi^2}\sum_{i=1,2}\int d^2p\ s_i\log\left(1-s_ie^{-\beta\mathcal{E}}\right)$$

[6] Assume N=1 and n: # of 5-branes.

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where $s_i = \pm$ and $\mathcal{E} = \sqrt{p_1^2 + p_2^2}$. Therefore,

$$2\pi \int d\mathcal{E} \log\left(\tanh\left(\frac{1}{2}\beta\mathcal{E}\right)\right)\mathcal{E} = -\frac{2\pi}{\beta^2}\frac{7}{4}\zeta(3),$$

$$S = \frac{21}{2\pi}N^2 V_2 T^2 \zeta(3)\left(p + q + \frac{N_F}{N}\right) + \mathcal{O}(\lambda).$$

For $p = q = 1$ and $N_F = 0$, ABJM result in [2] is reproduced. The power of N here, namely, N^2 deviates from $N^{\frac{3}{2}}$ derived from the gravity result. This problem remains unsolved because we are just using the gauge theory on M2-branes.

3.1 Matrix model free energy

Let us try another method to compute the entropy (or free energy) of SCFTs in large N limit with 't Hooft coupling $\lambda = N/k \ll 1$. We assume for simplicity NS5- and $(1,k)$5-branes are placed pairwise on the circle x^6 such that $p = q$ in Type IIB setup. One needs to compute the unitary matrix integral ($x = e^{-\beta}$) [25, 26, 28]:

$$\begin{aligned} Z = \int \prod_{I=1}^{2q} DU_I \exp \sum_{i=1}^{q}\sum_{n=1}^{\infty}\frac{1}{n}\Big(& z^{\text{unt}}(x^n)\big(\operatorname{Tr}(U_{2i}^n)\operatorname{Tr}(U_{2i+1}^{-n}) + (n \leftrightarrow -n)\big) \\ & + z^t(x^n)\big(\operatorname{Tr}(U_{2i-1}^n)\operatorname{Tr}(U_{2i}^{-n}) + (n \leftrightarrow -n)\big) \\ & + z^f_{2i}(x^n)\big(\operatorname{Tr}(U_{2i}^n) + \operatorname{Tr}(U_{2i}^{-n})\big) + z^f_{2i-1}(x^n)\big(\operatorname{Tr}(U_{2i-1}^n) \\ & + \operatorname{Tr}(U_{2i-1}^{-n})\big)\Big). \end{aligned} \tag{3.1}$$

Note that t, *unt* and f stand for twisted, untwisted and flavor, respectively. The matrix model arises from compactifying CSM theory on $S^1 \times S^2$ ($t \sim t + \beta$), taking suitable temporal gauge and integrating out matters. Here, Polyakov loop $U_I = e^{i\beta A_0^I}$ satisfies $U_{2q+1} = U_1$ and $U^{-1} = U^\dagger$.

By writing the measure as $DU = \prod_{n=1}^{\infty} d\rho_n \exp\left(-N^2 \sum_n \frac{\rho_n \rho_{-n}}{n}\right)$ with $\rho_n = \frac{1}{N}\operatorname{Tr} U^n$ (which facilitates large N limit), Z becomes

$$\begin{aligned} Z = \int \prod_{i,n} d\rho_{i,n} d\chi_{i,n} \exp \sum_{i,n} -\frac{N^2}{n}\Big(& \rho_{i,n}\rho_{i,-n} + \chi_{i,n}\chi_{i,-n} \\ & -\frac{1}{N}z^f_{2i,n}(\rho_{i,n} + \rho_{i,-n}) - \frac{1}{N}z^f_{2i-1,n}(\chi_{i,n} + \chi_{i,-n}) \\ & -z^{\text{unt}}_n\big(\chi_{i,n}\rho_{i+1,-n} + (n \leftrightarrow -n)\big) - z^t_n\big(\rho_{i,n}\chi_{i,-n} + (n \leftrightarrow -n)\big)\Big). \end{aligned}$$

Notice that ρ (χ) comes from the odd (even) subscript of U_I. Alternatively, in large N limit, one can introduce an eigenvalue density function for each U_I like

$$\sigma_I(\theta) = \frac{1}{2\pi} + \sum_{m=1}^{\infty}\frac{\rho_{I,m}}{\pi}\cos(m\theta), \qquad \int_0^{2\pi} d\theta \sigma_I(\theta) = 1$$

to solve the matrix model. Because U appears only in characters of $\mathrm{U}(N)$, one can just express in the diagonal form $U = diag(e^{i\theta_1}, \cdots, e^{i\theta_N})$ and get rid of irrelevant angular parts. Also, DU is the invariant Haar measure normalized as $\int DU\,\mathrm{Tr}_{R'}\,U\cdot\mathrm{Tr}_R\,U^\dagger = \delta_{R'R}$. It is easily confirmed that either way leads to the same expression of Z.

Further taking high temperature limit $\beta \ll 1$ to facilitate the comparison with gravity results, we find that there is a saddle point $\rho_{i,n} = \chi_{i,n} = 1$. Because of

$$z_n^t = z_n^{\rm unt} = z_n = 2\Big(z_B(x^n) + (-)^{n+1} z_F(x^n)\Big), \qquad z_{i,n}^f = N_F^i z_n,$$
$$z_B(x) = \frac{x^{\frac{1}{2}}(1+x)}{(1-x)^2}, \qquad z_F(x) = \frac{2x}{(1-x)^2},$$

by using asymptotics of z's

$$z_B(x^n) \to \frac{2}{(n\beta)^2} + \mathcal{O}\left(\frac{1}{n\beta}\right), \qquad z_F(x^n) \to \frac{2}{(n\beta)^2} + \mathcal{O}\left(\frac{1}{n\beta}\right)$$

and ζ-function,

$$Z \sim \exp \sum_{n=1}^{\infty} \frac{2N^2}{n}\left(\frac{N_F}{N} + 2q\right) z_n$$

gives rise to

$$F = -2T^3\zeta(3)\frac{7}{4}\mathcal{N}, \qquad S = \frac{21}{2}T^2\zeta(3)\mathcal{N}, \qquad \mathcal{N} = 4(NN_F + 2qN^2).$$

We find agreement with the previous result up to some irrelevant constant $V_2\pi/4\pi^2$.

4 Entropy from gravity dual

Now let us proceed to examine issues about the entropy (counting degree of freedom) of the obtained $\mathcal{N}$=3 SCFT using both the 11D M-theory dual and known IIA dual geometry with probe flavor branes embedded. The entropy obtained by making use of the free field theory approximation will therefore be compared with these gravity calculations.

To fulfill this purpose, we shall demonstrate more precisely the 8D transverse space to M2-branes is a 8D hyperKähler manifold, Eschenburg space, whose isometry, holonomy, and volume have been known (see also Introduction).

4.1 Eschenburg space as gravity dual

According to the remarkable work of Gauntlett, Gibbons, Papadopoulos and Townsend [22], one is able to have a dictionary translating certain IIB 5-brane configuration into a 11D M-theory geometry $R^{1,2} \times \mathbf{M}_8$. This works also in our case where $\mathbf{M}_8$ is now specified as Eschenburg space. As noted before, its hyperKähler structure makes the symmetry match with the newly obtained $\mathcal{N}$=3 SCFT quite successful.

For $\varphi_i \in (0, 4\pi]$,

$$\begin{aligned} ds_{8D}^2 &= \frac{1}{2}U_{ij}d\boldsymbol{x}_i\cdot d\boldsymbol{x}_j + \frac{1}{2}U^{ij}(d\varphi_i + A_i)(d\varphi_j + A_j), \\ A_i &= d\boldsymbol{x}_j\cdot\boldsymbol{\omega}_{ji} = dx_j^a\ \omega_{ji}^a, \qquad \partial_{x_j^a}\omega_{ki}^b - \partial_{x_k^b}\omega_{ji}^a = \epsilon^{abc}\partial_{x_j^c}U_{ki} \end{aligned} \tag{4.1}$$

where $i,j,k=1,2$ and $a,b,c=1,2,3$ (Cartesian label). Note that U_{ij} is a 2 by 2 symmetric matrix:

$$U_{ij}=\frac{1}{2}\begin{pmatrix}\dfrac{p}{|\boldsymbol{x}_1|}+\dfrac{q}{|\boldsymbol{x}_1+k\boldsymbol{x}_2|} & \dfrac{kq}{|\boldsymbol{x}_1+k\boldsymbol{x}_2|}\\ \dfrac{kq}{|\boldsymbol{x}_1+k\boldsymbol{x}_2|} & \dfrac{k^2q}{|\boldsymbol{x}_1+k\boldsymbol{x}_2|}\end{pmatrix} \tag{4.2}$$

in the case of p NS5- and q $(1,k)$5-branes on Type IIB side. Here, $\boldsymbol{x}_1=(345)$ and $\boldsymbol{x}_2=(789)$. The normalization of U is chosen such that it gives (4.11) after M2-brane backreaction.

Let us perform the following $GL(2)$ transformation

$$(\boldsymbol{x}'_1,\boldsymbol{x}'_2)=(\boldsymbol{x}_1,\boldsymbol{x}_2)G^t, \qquad (\varphi'_1,\varphi'_2)=(\varphi_1,\varphi_2)G^{-1},$$

$$G=\begin{pmatrix}p & 0\\ q & kq\end{pmatrix}, \qquad U\ \rightarrow\ U'=\frac{1}{2}\begin{pmatrix}\dfrac{1}{|\boldsymbol{x}'_1|} & 0\\ 0 & \dfrac{1}{|\boldsymbol{x}'_2|}\end{pmatrix}.$$

The effect of adding N_F flavors is to include $\Delta U=\frac{1}{2}\begin{pmatrix}0 & 0\\ 0 & \dfrac{N_F}{|\boldsymbol{x}_2|}\end{pmatrix}$ to U and thus

$$\Delta U'=\frac{1}{2}\begin{pmatrix}\dfrac{qN_F}{kp\not{L}} & \dfrac{-N_F}{k\not{L}}\\ \dfrac{-N_F}{k\not{L}} & \dfrac{pN_F}{kq\not{L}}\end{pmatrix}, \qquad \not{L}=|p\boldsymbol{x}'_2-q\boldsymbol{x}'_1|.$$

We see that due to non-zero N_F, $(\boldsymbol{x}'_1,\boldsymbol{x}'_2)$ should be rotated simultaneously by a common element of $\mathrm{SO}(3)_R$ in order to preserve $\not{L}$. Moreover, $\mathrm{U}(1)_b\times\mathrm{U}(1)_d$ corresponds to two U(1)'s of (φ'_1,φ'_2) which can be promoted to a local symmetry and offset by gauge transformations of (A'_1,A'_2). These together again agree with the above argument.

In order to make the structure of Eschenburg space, through $\boldsymbol{x}'_1\rightarrow-\boldsymbol{x}'_1$, $\varphi'_1\rightarrow-\varphi'_1$ and rewriting $\Delta U'$ as

$$\Delta U'=\frac{1}{2}\begin{pmatrix}\dfrac{t_1^2}{t_3|t_1\boldsymbol{x}'_1+t_2\boldsymbol{x}'_2|} & \dfrac{t_1t_2}{t_3|t_1\boldsymbol{x}'_1+t_2\boldsymbol{x}'_2|}\\ \dfrac{t_1t_2}{t_3|t_1\boldsymbol{x}'_1+t_2\boldsymbol{x}'_2|} & \dfrac{t_2^2}{t_3|t_1\boldsymbol{x}'_1+t_2\boldsymbol{x}'_2|}\end{pmatrix},$$

we then find that ds^2_{8D} in (4.1) leads to Eschenburg space labeled by three coprime natural numbers $(t_1,t_2,t_3)=(qN_F,pN_F,kpq)$.

When $t_1\neq t_2\neq t_3$, it has the least isometry $\mathrm{SU}(2)\times\mathrm{U}(1)^2$ and preserves a fraction 3/16 of 32 SUSY (defining feature of a cone over 7D tri-Sasakian manifolds).[7] According to [17], one has the following relation between 5- and 7-cycles inside Eschenburg space:

$$\frac{vol(S^5)}{vol(\Sigma_5)}=\frac{vol(S^7)}{vol(\tilde{\mathcal{M}}_7)}=\frac{(q+p)(N_F+kq)(N_F+kp)}{(N_F+k(q+p))}. \tag{4.3}$$

[7] An enhancement to a fraction 3/8 happens while one zooms into the near-horizon region of M2-branes, i.e. $R^{1,2}\times$ Cone$(\mathcal{B}_7)\rightarrow AdS_4\times\mathcal{B}_7$.

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4.2 Entropy from M-theory dual

Having said that the transverse geometry is a 8D hyperKähler cone, after the backreaction of M2-branes we are left with $AdS_4 \times \tilde{\mathcal{M}}_7$ under the normalization

$$6R^6 vol(\tilde{\mathcal{M}}_7) = (2\pi\ell_p)^6 N, \qquad R^6_{S^7} = 2^5\pi^2 N l_p^6. \tag{4.4}$$

Note that $R = 2R_{\rm AdS}$ is the radius of $\tilde{\mathcal{M}}_7$. This background will be taken as the cornerstone of studying the strongly coupled behavior of our $\mathcal{N}$=3 CSM theory. To count degrees of freedom via the above M-theory dual, we replace AdS_4 with AdS-Schwarzschild black hole and evaluate its Bekenstein-Hawking entropy.

AdS-Schwarzschild black hole metric is given by

$$ds^2 = \left(\frac{4r^2}{R^2} + 1 - \frac{M}{r}\right) d\tau^2 + \frac{dr^2}{\left(\frac{4r^2}{R^2} + 1 - \frac{M}{r}\right)} + r^2 d\Omega_2^2. \tag{4.5}$$

This metric can serve as a dual description of the finite-temperature CSM theory on $S^1 \times S^2$. (4.5) is smooth if the period of τ satisfies

$$\beta = \frac{\pi R^2 r_0}{3r_0^2 + \frac{R^2}{4}} \tag{4.6}$$

where r_0 is the horizon radius. Solving (4.6) in terms of β, we obtain

$$r_0 = \frac{\pi R^2}{6\beta} + \sqrt{\left(\frac{\pi R^2}{6\beta}\right)^2 - \frac{R^2}{12}}. \tag{4.7}$$

From (4.7), it is found that AdS black holes exist when $\beta < \pi R/\sqrt{3}$ and Hawking-Page phase transition [29] occurs at $\beta_c = \pi R/2$ above the temperature bound.

We can use Bekenstein-Hawking area law to yield the entropy per $\frac{1}{4} vol(S^2) R^2$:

$$\begin{aligned} S \;&\equiv\; \frac{2^{\frac{3}{2}}\pi^2 N^{\frac{3}{2}}}{27\beta^2}\left(1 + \sqrt{1 - \frac{3\beta^2}{\pi^2 R^2}}\right)^2 \sqrt{\frac{vol(S^7)}{vol(\tilde{\mathcal{M}}_7)}} \\ &\rightarrow \frac{2^{\frac{7}{2}}\pi^2 N^{\frac{3}{2}}}{27\beta^2}\sqrt{\frac{vol(S^7)}{vol(\tilde{\mathcal{M}}_7)}} \quad \text{(at high temperature).} \end{aligned} \tag{4.8}$$

Note that the unit volume of 7-cycle of Eschenburg space is related to that of S^7 via (4.3). See appendix A for another point of view on the derivation of S from GKP-W relation. By assuming $N_F \ll k$ ($\lambda \ll N/N_F$) and expanding (4.8) in powers of $1/k$, (4.8) looks like

$$\begin{aligned} S = \frac{2^{\frac{7}{2}}\pi^2 N^{\frac{3}{2}}}{27\beta^2}\Bigg[&\sqrt{kpq} + \frac{N_F\left(p^2+q^2+pq\right)}{2\sqrt{kpq}(p+q)} + \\ &+ \frac{N_F^2}{(kpq)^{\frac{3}{2}}}\left(-\frac{1}{2}\left(p^2+q^2\right) + \frac{3\left(p^2+q^2+pq\right)^2}{8\left(p+q\right)^2}\right) + \mathcal{O}\left(N_F^3 k^{-\frac{5}{2}}\right)\Bigg]. \end{aligned} \tag{4.9}$$

It is convenient to rewrite (4.9) in terms of 't Hooft coupling as

$$S = \frac{2^{\frac{7}{2}}\pi^2}{27\beta^2}\left[N^2\sqrt{\frac{pq}{\lambda}} + \frac{\sqrt{\lambda}NN_F\left(p^2+q^2+pq\right)}{2\sqrt{pq}(p+q)} \right. \\ \left. +\frac{\lambda^{\frac{3}{2}}N_F^2}{(pq)^{\frac{3}{2}}}\left(-\frac{1}{2}\left(p^2+q^2\right)+\frac{3\left(p^2+q^2+pq\right)^2}{8\left(p+q\right)^2}\right)\right]+\dots. \tag{4.10}$$

Setting $p = q = 1$ in (4.10), we recover results of the flavored ABJM theory in [8]. The 1st term on r.h.s. of (4.9) is the famous $N^{\frac{3}{2}}$ factor of M2-branes. The 2nd term can be interpreted as the tree-level effect of adding flavors as will be shown to be captured by IIA probe D6-branes. The 3rd term proportional to $\lambda^{\frac{3}{2}}N_F^2$ represents degrees of freedom from mesonic flavor states. Higher order terms may describe the interaction between flavor and bi-fundamental fields. We also find the 2nd term in (4.10) $\propto \sqrt{\lambda}NN_F$ has an additional $\sqrt{\lambda} \gg 1$ factor compared to weak coupling results. This may suggest that in strong coupling regime degrees of freedom due to flavors gets increasing quite a lot.

4.3 On-shell action of flavor D6-brane

We now turn to Type IIA viewpoint of evaluating the entropy. This involves treating flavors as probe branes in a given geometry. To discuss their on-shell action, one must first clarify how they are embedd.

Let us briefly describe the dual geometry of $\mathcal{N}$=4 SCFT [10] constructed via IIB N circular D3-, p NS5- and q $(1,k)$5-branes:

$$\begin{aligned} ds^2_{11D} &= \frac{R^2}{4}ds^2_{AdS_4} + R^2 ds^2_7, \qquad R = \ell_p(2^5Nkpq\pi^2)^{1/6}, \\ ds^2_7 &= d\xi^2 + \frac{1}{4}\cos^2\xi\Big((d\chi_1+\cos\theta_1 d\phi_1)^2 + d\theta_1^2 + \sin^2\theta_1 d\phi_1^2\Big) \\ &\quad + \frac{1}{4}\sin^2\xi\Big((d\chi_2+\cos\theta_2 d\phi_2)^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2\Big), \\ (\chi_1,\chi_2) &\sim \left(\chi_1+\frac{4\pi}{kp}, \chi_2+\frac{4\pi}{kq}\right) \sim \left(\chi_1+\frac{4\pi}{p},\chi_2\right), \\ 0 &< \xi \le \frac{\pi}{2}, \qquad 0 < \theta_i \le \pi, \qquad 0 < \phi_i \le 2\pi. \end{aligned} \tag{4.11}$$

Its isometry, two copies of SU(2)$\times$U(1), is easily read off because there are two (orbifolded) S^3 fibered over a segment $[0,1]$. It is straightforward to show that (4.11) is equivalent to (4.1) with (4.2) via including the near-horizon warp factor of M2-branes and changing variables as in [9].

For simplicity, we set $p = q$ and KK reduce to IIA string theory. Recall

$$ds^2_{11D} = e^{-\frac{2}{3}\Phi}ds^2_{\text{IIA}} + e^{\frac{4}{3}\Phi}(d\varphi+\cdots)^2,$$

then,

$$
\begin{aligned}
ds_7^2 &= ds_6^2 + \frac{1}{k^2q^2}(d\tilde{y} + \tilde{A})^2, \qquad e^{2\Phi} = \frac{R^3}{k^3q^3},\\
\tilde{A} &= kq\left(\frac{1}{2}\cos^2\xi(d\psi + \cos\theta_1 d\phi_1) + \frac{1}{2}\sin^2\xi\cos\theta_2 d\phi_2\right),\\
ds_6^2 &= d\xi^2 + \frac{1}{4}\cos^2\xi\sin^2\xi\Big(d\psi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2\Big)^2\\
&\quad + \frac{1}{4}\cos^2\xi(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2) + \frac{1}{4}\sin^2\xi(d\theta_2^2 + \sin^2\theta_2 d\phi_2^2),\\
ds_{\rm IIA}^2 &= L^2(ds_{AdS4_4}^2 + 4ds_6^2), \qquad L^2 = \frac{R^3}{4kq},\\
0 < y &= \frac{1}{kq}\tilde{y} \le \frac{2\pi}{kq}, \qquad 0 < \psi \le \frac{4\pi}{q}, \qquad \chi_1 = \psi + 2y, \qquad \chi_2 = 2y
\end{aligned}
\tag{4.12}
$$

with fluxes

$$
\begin{aligned}
F_2 &= kq\Big(\cos\xi\sin\xi d\xi \wedge (d\psi + \cos\theta_1 d\phi_1 - \cos\theta_2 d\phi_2)\\
&\quad -\frac{1}{2}\cos^2\xi\sin\theta_1 d\phi_1\wedge d\theta_1 - \frac{1}{2}\sin^2\xi\sin\theta_2 d\phi_2\wedge d\theta_2\Big) = -\frac{kq}{2L^2}\omega_2,\\
F_4 &= -\frac{3}{8}R^3\epsilon_{AdS_4}, \qquad \epsilon_{AdS_4} = r^2 dt\wedge dx^1\wedge dx^2\wedge dr.
\end{aligned}
\tag{4.13}
$$

It is obvious that $\mathcal{N}$=6 ABJM case differs from ours only by a factor q. As will be explained more in appendix A, by imposing $\boldsymbol{x}_2 = (789) = 0$ (locus of IIB flavor D5-branes) on (4.11), there then appears $AdS_4 \times S^3/\mathbf{Z}_{2q}$:

$$
\begin{aligned}
\psi' &= \psi/2, \qquad \theta_1 = \theta_2(=\theta), \qquad \phi_1 = -\phi_2(=\phi), \qquad \xi = \frac{\pi}{4},\\
ds_{S^3/Z_{2q}}^2 &= \frac{1}{4}\left(d\psi' + \cos\theta d\phi\right)^2 + \frac{1}{4}\left(d\theta^2 + \sin^2\theta d\phi^2\right).
\end{aligned}
\tag{4.14}
$$

Let us go to evaluate the on-shell action of probe D6-branes. The induced metric at finite temperature is

$$
ds_{D6}^2 = L^2\left(\frac{dr^2}{r^2\left(1-\left(\frac{r_0}{r}\right)^3\right)} - r^2\left(1-\left(\frac{r_0}{r}\right)^3\right)dt^2 + r^2 d\vec{x}^2 + 4ds_{S^3/Z_{2p}}^2\right), \tag{4.15}
$$

where Hawking temperature is given by $T = 3r_0/4\pi$. The on-shell action of N_F D6-branes per volume V_2 of $\vec{x} = (x^1, x^2)$ is

$$
I_{D6} = -\frac{2^3 N_F}{(2\pi)^6}e^{-\Phi}L^7 vol(S^3/\mathbf{Z}_{2p})\cdot\int dt\int_{r_0}^{\infty} dr\ r^2 \ \to\ \frac{2^{\frac{7}{2}}}{81}\sqrt{\lambda}\pi^2 N_F N T^2, \tag{4.16}
$$

where we have subtracted the divergent part at infinity. Note that I_{D6} does not depend on p (or q). From (4.16) the free energy and entropy per V_2 are

$$
F_{D6} = -TI_{D6} = -\frac{2^{\frac{7}{2}}}{81}\sqrt{\lambda}\pi^2 N_F N T^3, \qquad S_{D6} = \frac{2^{\frac{7}{2}}}{27}\sqrt{\lambda}\pi^2 N_F N T^2. \tag{4.17}
$$

Interestingly, we find that S_{D6} is again accompanied by a $\sqrt{\lambda}$ factor compared to zero-coupling approximation. In addition, S_{D6} is larger than the 2nd term in (4.9) by a factor $4/3$ if $p = q$! But we should be cautious because λ has different values in the two cases. In M-theory where M-circle ($\sim R/\ell_p kq \gg 1$) is decompactified, $\lambda \gg k^4 q^4$. Here, in IIA theory $g_s \ll 1$ means that $1 \ll \lambda \ll k^4 q^4$. It still seems interesting to pursue this 4/3 problem against the famous one in [15].

5 Superconformal index

Superconformal indices of 3D SCFTs are considered in [27, 30–33]. Let us compute that of our $\mathcal{N}$=3 SCFT containing flavors. Because the internal 7-manifold $\tilde{\mathcal{M}}_7$ is not homogeneous in general, to study KK spectra on $\tilde{\mathcal{M}}_7$ is quite difficult. As prescribed in [27],

$$I = \mathrm{Tr}(-)^F x^{\epsilon+j} y_1^{h_2} \cdots y_{M-1}^{h_M}$$

receives contributions from short multiplets. $M = [\mathcal{N}/2]$ gets related to its superconformal group $OSp(\mathcal{N}|4)$. Also, ϵ, j and h_i are eigenvalues of Cartan generators of bosonic subgroup $\mathrm{SO}(2) \times \mathrm{SO}(3) \times \mathrm{SO}(\mathcal{N})$ of $OSp(\mathcal{N}|4)$. For $\mathcal{N}$=3, I gets simplified to

$$I = \mathrm{Tr}(-)^F x^{\epsilon+j}. \tag{5.1}$$

Using an unitary matrix model prescribed in [26, 32], we can instead compute (5.1) by

$$\begin{aligned}
I &= \int \prod_{I=1}^{2q} DU_I \exp\left(\sum_R \sum_{n=1}^{\infty} \frac{1}{n} F_R(x^n)\chi_R(U_I^n)\right) \\
&= \int \prod_{I=1}^{2q} DU_I \exp \sum_{i=1}^{q} \sum_{n=1}^{\infty} \frac{1}{n}\Big(F_n^t\big(\mathrm{Tr}(U_{2i}^n)\,\mathrm{Tr}(U_{2i+1}^{-n}) + (n \leftrightarrow -n)\big) \\
&\qquad + F_n^{\mathrm{unt}}\big(\mathrm{Tr}(U_{2i-1}^n)\,\mathrm{Tr}(U_{2i}^{-n}) + (n \leftrightarrow -n)\big) \\
&\qquad + N_F^{2i} F_n^f\big(\mathrm{Tr}(U_{2i}^n) + \mathrm{Tr}(U_{2i}^{-n})\big) + N_F^{2i-1} F_n^f\big(\mathrm{Tr}(U_{2i-1}^n) + \mathrm{Tr}(U_{2i-1}^{-n})\big)\Big)
\end{aligned}$$

where all conventions about unitary matrices follow (3.1). Again, we assumed that NS5- and $(1,k)$5-branes are placed pairwise on the circle such that $p = q$. By using same techniques as in (3.1),

$$\begin{aligned}
I = \int \prod_{i,n} d\rho_{i,n} d\chi_{i,n} \exp \sum_{i,n} -\frac{N^2}{n}\Big(&|\rho_{i,n}|^2 + |\chi_{i,n}|^2 \\
&-\frac{1}{N} N_F^{2i} F_n^f(\rho_{i,n} + \rho_{i,-n}) - \frac{1}{N} N_F^{2i-1} F_n^f(\chi_{i,n} + \chi_{i,-n}) \\
&-F_n^{\mathrm{unt}}\big(\chi_{i,n}\rho_{i+1,-n} + (n \leftrightarrow -n)\big) - F_n^t\big(\rho_{i,n}\chi_{i,-n} + (n \leftrightarrow -n)\big)\Big)
\end{aligned}$$

with

$$F_n^{\mathrm{unt}} = F_n^t = F_n^f = F_n = F(x^n), \qquad F(x) = \frac{\sqrt{x}}{1+x}.$$

Further setting $N_F^i = m$ for simplicity and rewriting I as (M_n: $4q \times 4q$ matrix)

$$I = \int \prod_n dc_n \exp \sum_n -\frac{N^2}{2n}(c_n^t M_n c_n + p_n^t c_n + c_n^t p_n),$$
$$c_n^t = (\rho_{1,n}, \chi_{1,n}, \rho_{1,-n}, \chi_{1,-n}, \cdots), \qquad p_n = -\frac{2m}{N} F_n^f \mathbf{1}_{1\times q},$$

one soon performs this Gaussian integral to yield

$$I = \prod_{n=1}^{\infty} \frac{(1+x^n)^{2q}}{(1-x^{nq})^2} \cdot \exp \sum_n \mathcal{K}_n, \qquad \mathcal{K}_n = \frac{N^2}{2n} p_n^t M_n^{-1} p_n = \frac{2m^2}{n} F_n^2 \sum_{a,b} \left(M_n^{-1}\right)_{ab} \tag{5.2}$$

with

$$M_n = \begin{pmatrix} \ddots & & \mathcal{Q}_{4\times4}^T \\ & \mathcal{S}_{8\times8} & \\ \mathcal{Q}_{4\times4} & & \ddots \end{pmatrix}, \qquad \mathcal{S}_{8\times8} = \begin{pmatrix} \mathcal{R}_{4\times4} & \mathcal{Q}_{4\times4} \\ \mathcal{Q}_{4\times4}^T & \mathcal{R}_{4\times4} \end{pmatrix}. \tag{5.3}$$

Here, $\ddots$ denotes $\mathcal{S}$ and $\mathcal{Q}$ in the corner of M_n is the contribution from the periodic condition $\rho_{q+1,n} = \rho_{1,n}$. $\mathcal{Q}$ has only non-zero elements $\mathcal{Q}_{41} = \mathcal{Q}_{23} = -F_n$, while

$$\mathcal{R} = \begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P} & 0 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} 1 & -F_n \\ -F_n & 1 \end{pmatrix}.$$

To evaluate how the *flavor* sector contributes to I lies in expanding the exponential w.r.t. m. We leave the comparison with gravity results in future works.

6 Baryonic operator

Let us examine the correspondence concerning baryonic operators. If we assume $N \gg N_F$, baryons like $\epsilon^{i_1\cdots i_N} Q^I \cdots Q^I$ must be ruled out. Then, the possibility lies in

$$B^I = \epsilon_{j_1\cdots j_N} \epsilon^{i_1\cdots i_N} A^I \cdots A^I, \qquad B^J = \epsilon_{j_1\cdots j_N} \epsilon^{i_1\cdots i_N} A^J \cdots A^J \tag{6.1}$$

where $\mathrm{SU}(2)_R$, color and flavor indices are suppressed, while I (J) stands for twisted (untwisted) hypermultiplets. Their conformal dimensions can be determined from the superpotential $\mathcal{L}'_{\text{pot}}$ to be $\Delta(B) = N/2$. On the gravity side, Δ can be confirmed via M5-branes wrapping 5-cycles Σ_5 inside Eschenburg space. Upon using (4.3) and (4.4),

$$\Delta = R_{\text{AdS}} \cdot m_{M5} = \frac{1}{2} \tau_{M5} R^6 vol(\Sigma_5) = \frac{\pi N}{6} \frac{vol(\Sigma_5)}{vol(\tilde{\mathcal{M}}_7)} = \frac{N}{2}$$

for large M5-brane mass. We see that Δ is independent of (t_1, t_2, t_3) as pointed out in [17].

When it comes to degeneracy, both di-baryons above having $N+1$ degeneracy form a spin $N/2$ rep. of $\mathrm{SU}(2)_R$. From

$$\frac{vol(\tilde{\mathcal{M}}_7)}{vol(\Sigma_5)} = \frac{vol(S^7)}{vol(S^5)} = \frac{\pi}{3} \sim \frac{1}{12} vol(S^2),$$

we can think that topologically what is transverse to an M5-brane inside $\tilde{\mathcal{M}}_7$ is roughly a 2-sphere such that the argument similar to [34] is still applicable. That is, the degeneracy of di-baryons is accounted for by N units of 7-form flux penetrating S^2. Collective coordinates of an M5-brane thus behave quantum mechanically as if there were $N+1$ degenerated states in the lowest Landau level under N units of magnetic flux through S^2.

Finally, we comment on how many independent di-baryons are there. According to [10] without flavors, the decomposability of $\prod_I B^I$ and $\prod_J B^J$ (dressed by appropriate monopole operators) into mesons gives rise to totally $p+q-2$ independent di-baryons. A detailed survey on the homology $H_5(\mathcal{M}_7, \mathbf{Z}) = \mathbf{Z}^{p+q-2}$ reveals the same thing.[8] These can be put another way, i.e. gauge-variant di-baryons are charged under $p+q-2$ U(1) gauge fields except for the two (diagonal one and dual photon) which are not involved in performing a hyperKähler quotient as said in section 1. Naively, this means that the RR 6-form potential should therefore be expanded like $C_6 \sim \sum_{I=1}^{p+q-2} \omega_5^I \wedge A^I$. In other words, $H_5(\mathcal{M}_7, \mathbf{Z}) = \mathbf{Z}^{p+q-2}$ (ω_5: volume form of 5-cycle).

In our case, given Betti numbers

$$b_2(\tilde{\mathcal{M}}_7) = b_5(\tilde{\mathcal{M}}_7) = 1,$$

it seems there is only one single di-baryon though. In view of (6.1), it seems there should be more independent di-baryons according to arguments given above. We wish to resolve the discrepancy in a future work.

7 Conclusion

In this note we provide a gravity dual for the flavored $\mathcal{N}$=4 Chern-Simons-matter theory which is a kind of $\mathcal{N}$=3 SCFT. From the following three viewpoints:

1. SUSY and global symmetry match
2. hyperKähler quotient construction of the moduli space
3. GGPT method of identifying the M-theory transverse geometry from the given Type IIB 5-brane configuration

we get confident in regarding our proposed Eschenburg space as an adequate candidate.

To study further the correspondence between both, we go to count degrees of freedom. On the field theory side, this is done by taking large N zero-coupling approximation such that an unitary matrix model previously known fulfills our purpose. On the gravity side, two approaches are tried, namely, we calculate the entropy from both the 11D AdS-Schwarzschild-Eschenburg black hole and an on-shell action of probe D6-branes in Type IIA geometry which is dual to $\mathcal{N}$=4 CSM. It is seen that field theory results are corrected by multiplying a factor $\sqrt{\lambda}$ to NN_F terms. This suggests that in strong coupling regime degrees of freedom due to adding flavors increase extremely. We also study gravity duals of mesonic and baryonic operators and find agreements on their conformal dimensions, and so on.

[8] See [35] for detailed considerations about homology in AdS$_4$/CFT$_3$.

Moreover, an $\mathcal{N}$=3 superconformal index is computed, though a comparison with the one from gravity is left in a future work due to essential difficulties in deriving Kaluza-Klein spectra on inhomogeneous $\tilde{\mathcal{M}}_7$. Nevertheless, for $(t_1, t_2, t_3) = (1, 1, 1)$, i.e. $\tilde{\mathcal{M}}_7 = N(1, 1)$, its KK spectra are known in some literature [36–38] and hence the comparison seems worthy of trying. Since Eschenburg space metric has not yet been fully exploited, we wish to report progress towards its application soon.

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A Degrees of freedom in 3D $\mathcal{N} = 3$ SCFT

We can roughly evaluate the degrees of freedom of the strongly coupled 3D SCFT via GKP-W relation [23, 24].[9]

We just compute the correlation function of two energy-momentum tensors

$$\langle T(x)T(y)\rangle = \frac{\delta^2 S_{\text{gravity}}}{\delta h \delta h} \sim \frac{c}{|x-y|^{2\Delta}}, \qquad \Delta = 3$$

where h is the metric perturbation around AdS_4 boundary, while c may contain the information about degrees of freedom in the 3D SCFT. Because c is dimensionless and (G_D: Newton constant)

$$S_{\text{gravity}} = \frac{1}{G_4} \int d^4x \sqrt{-g}(\mathcal{R} - \Lambda),$$

the only choice for c is

$$c \sim \frac{R_{\text{AdS}}^2}{G_4}, \qquad G_4 = \frac{G_{11}}{R^7 vol(\tilde{\mathcal{M}}_7)},$$
$$R = 2R_{\text{AdS}}, \qquad G_{11} = (2\pi)^8 \ell_p^9, \qquad \to c \sim \frac{N^{\frac{3}{2}}}{\sqrt{vol(\tilde{\mathcal{M}}_7)}}$$

where R is the radius of $\tilde{\mathcal{M}}_7$ and we have used (4.4). As is shown in section 4, c is exactly what is computed via Bekenstein-Hawking area law via its M-theory dual.

B Taub-NUT space

In this appendix, we show that probe D6-branes wrap 012 plus 4D Taub-NUT space inside the 7D cone in IIA theory.

[9] This part is inspired by the lecture note of Yosuke Imamura.

Recall that flavor D5-branes occupying (012345) are localized at $\boldsymbol{x}_2 = (789) = 0$. Regarding this constraint as a moment map (at zero level set), we can further perform a hyperKähler quotient via rearranging ds^2_{8D} into $\left(\varphi_2\text{: M-circle, } U = \frac{1}{2}\tilde{U}\right)$

$$ds^2_{8D} = \frac{1}{4}\left(\tilde{U}_{ij}d\boldsymbol{x}_i \cdot d\boldsymbol{x}_j + \frac{4}{\tilde{U}_{11}}(d\varphi_1 + A_1)^2\right) + \frac{\tilde{U}_{11}}{\det \tilde{U}}\left(d\varphi_2 - \frac{\tilde{U}_{12}}{\tilde{U}_{11}}d\varphi_1\right)^2. \tag{B.1}$$

A rescale is done to get a period 2π M-circle. Imposing $\boldsymbol{x}_2 = 0$ and throwing away the last term, we have ($p = q$, $\boldsymbol{x}_1 = \boldsymbol{\rho}$, $\rho = |\boldsymbol{\rho}|$, $\varphi_1 = \psi \in (0, 4\pi]$)

$$\begin{aligned} ds^2_4 &= \frac{1}{2}\left(\frac{q}{\rho}d\boldsymbol{\rho}^2 + \frac{\rho}{q}(d\psi + A_1)^2\right), \\ A_1 &\to 2q\frac{-\rho^1 d\rho^2 + \rho^2 d\rho^1}{\rho(\rho + \rho^3)} = -2q\boldsymbol{\omega}\cdot d\boldsymbol{\rho}, \qquad \nabla \times \boldsymbol{\omega} = -\nabla\frac{1}{\rho}, \end{aligned} \tag{B.2}$$

which represents a multi-centered Taub-NUT whose $2q$ NUTs coincide. Owing to the cone structure, making M2-branes backreact and taking near-horizon limit, we have constant dilaton field

$$\begin{aligned} e^{\frac{4}{3}\Phi} &= H^{\frac{1}{3}}(r^2 = |\boldsymbol{x}_1'| + |\boldsymbol{x}_2'|)\cdot\left(\frac{\tilde{U}_{11}}{\det \tilde{U}}\right) = \text{const.}, \\ H &= 1 + \frac{\ell_p^6 2^5 N' \pi^2}{r^6} \sim \left(\frac{R^2}{r^2}\right)^3 \qquad r \to 0, \end{aligned}$$

which promises an AdS factor. Therefore, one can finally arrive at flavor D6-branes with worldvolume $AdS_4 \times S^3/\mathbf{Z}_{2q}$:

$$ds^2_{D6} = L^2\left(ds^2_{AdS_4} + 4ds^2_{S^3/Z_{2q}}\right). \tag{B.3}$$

On the other hand, if the level set is non-zero $\boldsymbol{x}_2 = \xi \neq 0$, i.e. adding massive fundamental flavors, it is readily seen that all NUTs will not coincide and TN_4 is partially resolved.

C Meson spectrum

Here, we consider mason spectra from Type IIA geometry in (4.12). This involves a D6-brane embedding with worldvolume action described by ($\ell_s = 1$)

$$S_{D6} = -T_{D6}\int d^7x\,\sqrt{-\det(g_{ab} + B_{ab} + 2\pi F_{ab})} - T_{D6}\int e^{2\pi F + B} \wedge \sum_p C_p. \tag{C.1}$$

We take the static gauge such that its worldvolume is parameterized by $(t, x, y, r, \theta, \phi, \psi')$ with $\theta = \theta_1 = \theta_2$ and $\phi = \phi_1 = -\phi_2$. The scalar perturbation on a stack of D6-branes concerning meson spectra is $\delta\xi = \eta = \rho(r)e^{ip\cdot x}Y_\ell(\Omega)$. Its angular part can be expanded by spherical harmonics on S^3:

$$\nabla^2 Y_\ell(\Omega) = -\ell(\ell + 2)Y_\ell(\Omega). \tag{C.2}$$

Let us assume in Type IIB picture there are totally F flavor D5-branes distributed over $2q$ intervals of x^6 as $\left(F_1, \cdots, F_{2q}\right)$ $(F = \sum_{I=1}^{2q} F_I)$ such that the flavor symmetry gets broken like $\mathrm{U}(F) \to \prod_I \mathrm{U}(F_I)$. Note that via T-dualizing x^6 to IIA the above information is encoded in the following holonomy (Wilson loop)

$$\exp\left(i \oint A_{\psi'} d\psi'\right) = \bigoplus_{I=1}^{2q} \omega^I \mathbf{1}_{F_I} \tag{C.3}$$

on F D6-branes with $\omega = \exp(2\pi i/2q)$ due to $\pi_1(S^3/\mathbf{Z}_{2q}) = \mathbf{Z}_{2q}$ in (B.3).

Moreover, spherical harmonics can be labeled by $Y_\ell^{IJ}(\Omega)$, i.e. the open string scalar mode η can be either of adjoint rep. $(I = J)$ or bi-fundamental rep. $(I \neq J)$ w.r.t. flavor groups depending on on which two stacks of D6-branes its ends are. Because only modes surviving the projection $\Gamma = \omega^{I-J} \exp \dfrac{4\pi i J_L^3}{2q}$ $(J_L \in \mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ of S^3 and ω^{I-J} stands for the acquired holonomy) remain, therefore

$$2m_L + I - J \in 2q\mathbf{Z}. \tag{C.4}$$

For Y_ℓ of $(m_L, m_R) = (\ell/2, \ell/2)$, this implies

$$\ell = k + 2q\mathbf{Z}, \qquad k = 0, \cdots, 2q-1. \tag{C.5}$$

Furthermore, due to [9]

$$\Delta = \frac{d}{2} \pm \sqrt{\left(\frac{d}{2}\right)^2 + m_\eta^2}, \qquad m_\eta^2 = \frac{\ell(\ell+2) - 8}{4},$$

one has

$$\Delta = 1 - \frac{\ell}{2} = 1 + \frac{k}{2} + q\mathbf{Z}. \tag{C.6}$$

From the superpotential in section 2, all A, B, Q and $\tilde{Q}$ have the same conformal dimension $1/2$, thus dual mesonic operators of Δ are like

$$\tilde{Q}^{I-1} A^I (A^I B^I)^{x_I} \cdots A^J (A^J B^J)^{x_J} Q^{J+1}, \qquad \sum_K x_K = q\mathbf{Z}. \tag{C.7}$$

As a remark, it is seen that meson spectra are not effected by gauge group ranks on different intervals. This can be seen from (C.1) where D6-brane DBI action has zero pull-back of NSNS 2-form flux $\propto \omega_2$ (Kähler form) arising from fractional M2-branes [39].

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